Infinite Dimensional Dynamical Systems
MATH580 PDEs I

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0 Introduction

Dynamical systems study long-term behaviour of the time evolution of systems from a geometrical perspective. In the finite-dimensional case of ODEs, existence of solutions for all times is a relatively easy question to answer, whereas certain PDEs can give rise to comparatively restrictive existence and uniqueness theory. For example, recall the Cauchy-Kovalevskaya Theorem, which only gives a positive radius of convergence of a power series solution, with no additional information on continuity with respect to initial conditions, nor anything about larger regions of existence. Studying PDEs with a dynamical lens will therefore require some work, but most importantly choosing an appropriate equation to analyze. To this effect, we focus on a reaction-diffusion model, an example of a dissipative parabolic PDE.

The structure of these notes is as follows: the first section presents a quick overview of the ideas we need from dynamical systems, with few proofs but some intuition and motivation for the approach we take; the second section is a discussion on Sobolev spaces and the presentation of useful results, with again few proofs, building up to the main part of these notes: the third section presents a proof of existence and uniqueness of a reaction-diffusion equation, with a polynomial reaction term with odd degree, done through Galerkin projections and energy estimates to obtain weak convergence to a solution, mostly following §III.1.1 of [9]. We can then get information on our global attractor, and an expository discussion is given on behaviour of solutions near and on the attractor.

Some of the notation used throughout the notes is common to [7] and [9], the 2 main sources used, but it is not standard so is summarized in Appendix A. The remaining appendices contain proofs of lemmas which are mostly analytical and not instructive enough to be included in the main body.

1 Dynamical systems

1.1 Attractors

A central object in dynamical systems theory is the evolution operator. Suppose our phase space is some metric space (H, d), then we study a family of operators $S(t)$ satisfying *semigroup* properties

$$\begin{aligned} S(t+s) &= S(s)S(t) \quad \forall s,t \geq 0, \\ S(0) &= I. \end{aligned}$$

Here t is thought of as time, so the 2 properties are quite intuitive: the first says passing time $t+s$ is the same as passing time t then time s, and the second says passing no time at all doesn't change anything. We also assume continuity of $S(t) : H \to H$ for every $t \geq 0$ for this section. Another property of the semigroup we will need is *uniform compactness*: we say the operators $S(t)$ are uniformly compact for t large if for every bounded set $B \subset H$ there exists $t_0 = t_0(B)$ such that

$$\bigcup_{t \geq t_0} S(t)B$$

is relatively compact, i.e. has compact closure. Essentially, passing enough time brings the dynamics into a compact set, the closest analogue to finiteness in topology[1]. Other objects of interest are then *orbits* under the semigroup, that is

$$\bigcup_{t \geq 0} S(t)u_0$$

[1] Apart from finiteness itself, of course.

for $u_0 \in H$, where one imagines this as a 1-parameter curve in our space H. More importantly, we are interested in long-term behaviour of solutions, encoded in the ω-limit set of points $u_0 \in H$, or sets $B \subset H$

$$\omega(u_0) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)u_0}, \quad \omega(B) = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} S(t)B}.$$

For a point $u_0 \in H$, $\omega(u_0)$ is the set of points that the orbit $\{u(t) = S(t)u_0 : t \geq 0\}$ approaches infinitely many times in forward time, and similarly for $\omega(B)$. These sets are easily proved to be invariant under the semigroup, where a set $A \subset H$ is *invariant* under $S(t)$ if $S(t)A = A$ for all $t \geq 0$. We can now define our main dynamical object of interest.

Definition 1. An attractor is a set $\mathcal{A} \subset H$ satisfying

1. $\mathcal{A}$ is invariant, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$,
2. $\mathcal{A}$ attracts trajectories from an open neighbourhood U as $t \to \infty$: for each $u_0 \in U$,

$$\operatorname{dist}(S(t)u_0, \mathcal{A}) \to 0 \text{ as } t \to \infty,$$

where $\operatorname{dist}(x, Y) = \inf_{y\in Y} d(x, y)$.

Associated to each attractor we can look at the largest neighbourhood U it attracts, called its basin of attraction. We can also define a *global attractor* $\mathcal{A}$, attracting all bounded sets of H, which then forms its basin of attraction. This set is unique and is therefore an object of clear importance to studying a dynamical system. Defining a global attractor in a consistent and intuitive way has required a lot of conceptual work and many different attempts, the reader is recommended [6] for an interesting discussion on this topic.

One usually finds an attractor after finding some absorbing set: a set B which all bounded sets enter after some time.

Definition 2. Let $B \subset H$ and let U be open with $B \subset U$. We say that B is *absorbing* in U if

$$\forall B_0 \subset U \text{ bounded }, \exists t_*(B) \text{ such that } S(t)B_0 \subset B \quad \forall t \geq t_*(B).$$

The following theorem gives conditions to check for existence of an attractor within some absorbing set.

Theorem 1. Let H be a metric space and let the operators $S(t)$ satisfy the semigroup property, be continuous for all $t \geq 0$ and be uniformly compact for large t. Additionally assume that $U \subset H$ is open and let $B \subset U$ be bounded and absorbing in U. Then the ω-limit set $\mathcal{A} = \omega(B)$ is a *compact attractor* attracting all bounded subsets of U. It is the maximal bounded attractor in U, in the sense that it contains every bounded attractor in U. Furthermore, if H is Banach, U is convex and $t \mapsto S(t)u_0$ is continuous from $\mathbb{R}_+$ to H for every $u_0 \in H$, then $\mathcal{A}$ is connected.

Note that $U = H$ gives a global attractor $\mathcal{A}$. The proof is omitted here, but can be found in [9, p. 23].

1.2 Classical and uniform Grönwall lemma

Grönwall-type bounds are a very common tool used in differential equations analysis, allowing control over the growth of solutions. We will require 2 such bounds later on: the first more classical, covered by any introductory course on theoretical ODEs, and the second a uniform bound over all time, at the cost of more restrictive conditions. The proofs are quite straightforward once one has the statement, and are both included in Appendix B.

Lemma 1 (Classical Grönwall)**.** Let g, h, y be locally integrable on (t_0, ∞), with $\frac{dy}{dt}$ locally integrable on (t_0, ∞), satisfying

$$\frac{dy}{dt} \leq gy + h \quad \text{for } t \geq t_0.$$

Then

$$y(t) \leq y(t_0) \exp\left(\int_{t_0}^{t} g(\tau)\, d\tau\right) + \int_{t_0}^{t} h(s) \exp\left(\int_{s}^{t} g(\tau)\, d\tau\right) ds \quad \text{for } t \geq t_0.$$

Lemma 2 (Uniform Grönwall)**.** Let g, h, y be positive locally integrable on (t_0, ∞) with $\frac{dy}{dt}$ locally integrable on (t_0, ∞), satisfying

$$\frac{dy}{dt} \le gy + h \quad \text{for } t \ge t_0,$$

as well as the following bounds for some $r, a_1, a_2, a_3 > 0$:

$$\int_t^{t+r} g(s)ds \le a_1, \quad \int_t^{t+r} h(s)ds \le a_2, \quad \int_t^{t+r} y(s)ds \le a_3, \quad \text{for } t \ge t_0.$$

Then

$$y(t+r) \le \left(\frac{a_3}{r} + a_2\right)\exp(a_1) \quad \text{for } t \ge t_0.$$

2 Sobolev spaces

2.1 Recall

The natural spaces for PDEs analysis are Sobolev spaces, the extension of L^p spaces that also requires decay on derivatives. However most L^p functions are not continuous[2], even less differentiable, at least not in the classical sense. We must therefore find a weaker notion of derivative, where we instead use integration by parts to pass on the derivative to a smooth function. In all of the following we have $\Omega \subset \mathbb{R}^n$ open and bounded, and we define $S = \partial\Omega$, which we require to be at least C^1. Let $C_c^\infty(\Omega)$ be the set of smooth compactly supported functions on Ω, known as *test functions*, which vanish on the boundary since Ω is open. We will focus on $p = 2$, since the Hilbert space structure unique to L^2 will be instrumental in the approach we take.

Definition 3. Let $u, v \in L^2(\Omega)$ and α be a multi-index. Then v is the α-th weak derivative of u, $v = D^\alpha u$ if

$$\int_\Omega v\,\varphi\,dx = (-1)^{|\alpha|}\int_\Omega u\,\partial^\alpha\varphi\,dx$$

for all test functions $\varphi \in C_c^\infty(\Omega)$.

Note that a distributional derivative might not always exist, cf. [3, p. 244]. Moreover, even if it does exist, we have no indication of its behaviour, e.g. decay. We hence define Sobolev spaces:

Definition 4. The *Sobolev space* $H^k(\Omega)$ is the set of all $u \in L^2(\Omega)$ such that $D^\alpha u$ exists and is in L^2, for all multi-indices α with $|\alpha| \le k$. The inner product on $H^k(\Omega)$ is

$$(u, v)_{H^k(\Omega)} = \sum_{|\alpha|\le k}(D^\alpha u, D^\alpha v),$$

where $(\cdot,\cdot)$ denotes the standard L^2 inner product. This then defines a norm for which $H^k(\Omega)$ is complete. We also define the space $H_0^k(\Omega)$ to be the closure of $C_c^\infty(\Omega)$ in H^k, and their dual spaces $H^{-k}(\Omega) = (H_0^k(\Omega))^*$.

Note that dependence on Ω will be omitted in the following unless explicitly mentioned otherwise, that is $H^k = H^k(\Omega)$. The space H_0^1 can be thought of as H^1 functions vanishing on the boundary S, and this notion is made rigorous by the trace operator, cf. [3, pp. 258-216]. Before moving onto our equation of interest, we need to make a few stops in the world of Sobolev spaces, most of which we will not prove.

2.2 Compactness and Poincaré inequality

The first result we need is a classic embedding theorem.

Theorem 2 (Rellich-Kondrachov compactness)**.** The Sobolev space H_0^1 is continuously and compactly embedded in L^2:

$$H_0^1 \hookrightarrow L^2.$$

[2]Or rather, most equivalence classes in L^p spaces do not contain any continuous functions.

By a continuous compact embedding, we mean that the inclusion map $i : H_0^1 \to L^2$ is continuous and compact, mapping bounded sets to pre-compact sets, when taking the H^1-norm on H_0^1 and the L^2-norm on L^2. Note that a much more general statement about higher order k of weak derivatives and different exponents p of integrability, and interestingly the dimension n of the domain Ω comes into play in the general case. The proof of this is quite involved, using mollifiers to work with smooth functions, the Gagliardo-Nirenberg-Sobolev inequality and the Arzelà-Ascoli theorem, c.f. [3, §5.7][3].

Not only are different Sobolev spaces different sets, but another aspect that sets them apart is the structure inherited by their norm. Indeed as we saw above, the H^1-norm on H_0^1 renders the inclusion map into L^2 compact, an obviously insane statement if we took the standard L^2-norm on H_0^1 instead. In the case of H_0^1, we can in fact choose an equivalent norm using the following result.

Proposition 1 (Poincaré Inequality)**.** *Let $u \in H_0^1$, then*

$$\|u\|_{L^2} \leq c_0 \|\nabla u\|_{L^2}$$

for some constant $c_0 = c_0(\Omega)$.

This result holds for more general Ω: it need only be bounded in one direction. Now observe that the H^1-norm has

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \leq (c_0^2 + 1)\|\nabla u\|_{L^2}^2,$$

such that the H^1 norm is equivalent to the norm $\|u\| = \|\nabla u\|_{L^2}$. We will therefore consider the inner product

$$((u, v)) = \sum_{i=1}^{n} (\partial_i u, \partial_i v)$$

and the norm

$$\|u\| = \{((u, u))\}^{1/2}$$

on H_0^1, notation we will use in what follows. Similarly we will use the notation $|u| = \|u\|_{L^2}$ and define $V = H_0^1$ and $H = L^2$ to stay consistent with Temam and Robinson. In a similar way there exists a constant $c_1 = c_1(\Omega) > 0$ such that

$$\|u\| \leq c_1 |\Delta u|, \tag{1}$$

for $u \in H_0^1 \cap H^2$, yielding an equivalence of norms on $H^2 \cap H_0^1$ between the usual H^2 norm and $|\Delta u|$.

2.3 Bilinear forms and Lax-Milgram

Consider a continuous linear operator $A : V \to V^*$, mapping into its dual space[4]. Then this defines a form $a : V \times V \to \mathbb{R}$ by $a(u, v) = \langle Au, v\rangle$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing: applying $Au \in V^*$ to $v \in V$, $\langle Au, v\rangle = Au(v)$. We have continuity and bilinearity of a from continuity and linearity of A and $\langle \cdot, \cdot \rangle$. Similarly given a continuous bilinear form $a : V \times V \to \mathbb{R}$, we can construct a linear operator $A : V \to V^*$: let $u \in V$, consider the linear functional $f_u \in V^*$ defined by $f_u(v) = a(u, v)$, and define $Au := f_u$. Linearity of A follows from linearity of a in the first component, and continuity follows from continuity of a. We can therefore associate a continuous linear operator $V \to V^*$ to any continuous bilinear form $V \times V \to \mathbb{R}$ and vice-versa. Moreover, in the case where a is coercive:

$$\exists \alpha \text{ s.t. } a(u, u) \geq \alpha \|u\|^2 \; \forall u \in V,$$

we have the Lax-Milgram Theorem:

Theorem 3. If a is a bilinear continuous coercive form on $V \times V^*$, then A is an *isomorphism* from V onto V^*.

One has a couple of ways of proving this: Evans does this by explicitly checking injectivity, checking the range $R(A) = V$ for surjectivity and explicitly checking uniqueness [3], while Temam uses a Contraction Mapping argument [9], which we go through here.

[3]This is where we require S being C^1.

[4]This discussion also applies to a general Hilbert space V.

Proof. If we can solve the equation

$$Au = f,$$

uniquely for all $f \in V^*$, then A is a bijection. We can rephrase $Au = f$ as the zero finding problem $Au - f = 0$, which we would like to solve using Newton's iterative method. Our Newton iteration map would look like

$$T(u) = u - D(Au - f)^{-1}(Au - f) = u - A^{-1}(Au - f),$$

but computing A^{-1} would require circular reasoning. Instead consider the canonical isomorphism $\Lambda : V \to V^*$ given by $\langle \Lambda u, v \rangle = ((u, v))$, the inner product in V, and replace A^{-1} by $\rho\Lambda^{-1}$ for some $\rho > 0$ to be determined. We show that

$$T(u) = u - \rho\Lambda^{-1}(Au - f)$$

is a contraction: let $u_1, u_2 \in V$, then

$$\begin{aligned}
\|Tu_1 - Tu_2\|^2 &= \|u_1 - u_2 - \rho\Lambda^{-1}(Au_1 - Au_2)\|^2 \\
&= \|u_1 - u_2\|^2 + \rho^2\|\Lambda^{-1}(Au_1 - Au_2)\|^2 - 2\rho((u_1 - u_2, \Lambda^{-1}A(u_1 - u_2))) \\
&= \|u_1 - u_2\|^2 + \rho^2\|(Au_1 - Au_2)\|_*^2 - 2\rho a(u_1 - u_2, u_1 - u_2) \\
&\leq (1 + \rho^2 M^2 - 2\rho\alpha)\|u_1 - u_2\|^2,
\end{aligned}$$

where $\|\cdot\|_*$ is the standard norm on the dual space V^*, essentially an operator norm. Here we used $\|A\|_{\text{op}} = M$ being the operator norm of A, finite since A is continuous, as well as coercivity of a for the last term. We can then choose $\rho < 2\alpha/M^2$ to make T a contraction. Then by the Contraction Mapping Theorem — or the Banach Fixed Point Theorem — the equation $Au = f$ has a unique solution $u \in V$ for all $f \in V^*$, thus defining A^{-1}. To show continuity, let $u = A^{-1}f$, then

$$\alpha\|u\|^2 \leq a(u, u) = \langle Au, u \rangle = \langle f, u \rangle \leq \|f\|_*\|u\|,$$

where we used coercivity of a in the first inequality and Cauchy-Schartz in the last inequality. This then gives

$$\|A^{-1}f\| = \|u\| \leq \frac{1}{\alpha}\|f\|_*,$$

so that $\|A^{-1}\|_{\text{op}} \leq \frac{1}{\alpha}$ and A^{-1} is continuous, thus ending the proof. □

Now consider the bilinear form

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$$

on V. This is exactly the V inner product $((\cdot, \cdot))$, so coercivity, continuity and bilinearity are obvious. We can then apply Lax-Milgram to a and obtain an operator $A : V \to V^*$ such that $a(u, v) = \langle Au, v \rangle$[5]. But what does A correspond to? Suppose for simplicity that $u, v \in C_c^\infty$, then

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega \Delta u \cdot v \, dx,$$

by Green's theorem, where the boundary term vanishes since u and v are compactly supported. Thus, $a(u, v) = \langle -\Delta u, v \rangle$ and $A = -\Delta$, at least on C_c^∞.

Finding that the Laplacian defines an isomorphism between 2 Hilbert spaces might seem strange. The most striking aspect is simply its boundedness, indeed differential operator are usually unbounded. The crucial detail here is the choice of domain and codomain equipped with their specific norms. We have defined $A : V \to V^*$, but we could restrict the codomain to $H \subset V^*$, which then forces us to modify the domain to

$$D(A) = \{v \in V : Av \in H\},$$

and the "new" operator is now $A : D(A) \to H$. We still have $A = -\Delta$, but changing the domain and codomain has possibly made A unbounded, and as we will see when looking at the Laplacian's spectrum, it well and truly is an unbounded operator given this choice.

[5]In fact Lax-Milgram is not needed here since $a(\cdot, \cdot)$ is exactly the H inner product and H is Hilbert, so the Riesz Representation Theorem already tells us it defines an isomorphism between V and V^*.

2.4 Spectral theorem

As in finite dimensions, much of the information contained in a linear operator is encoded in its eigenvalues and eigenvectors. In the infinite-dimensional case their existence is non-trivial, but we focus on self-adjoint compact operators, the closest analogues to finite-dimensional linear operators. Indeed, we have the following Spectral Theorem by Riesz, from Theorem 7.1 in Chap.VII §7 of [2]:

Theorem 4. Let V be an infinite-dimensional Hilbert space and $A : V \to V$ a compact, self-adjoint operator. Then A has an orthonormal basis of eigenvectors $\{v_n\}_{n\in\mathbb{N}}$ with associated eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ all non-negative, with $\lambda_n \to 0$ as $n \to \infty$.

The idea is to apply this to $A^{-1} = -\Delta^{-1}$ obtained above, then to invert the eigenvalues to get an orthonormal basis for $-\Delta$ itself. The first observation is that $A^{-1} : V^* \to V$ being continuous gives continuity of $A^{-1} : H \to V$. Then letting $i : V \hookrightarrow H$ be the continuous injection of V into H, we can see A^{-1} as a linear operator $H \to H$ (really defined by $i \circ A^{-1}$). For the second observation, note that i is compact by the Kondrachov Compactness Theorem 2. Then for a bounded set $B \subset H$, $A^{-1}(B) \subset V$ is bounded since A^{-1} is continuous, then $(i \circ A^{-1})(B) \subset H$ is pre-compact, so that

$$A^{-1} : H \to H \text{ is a compact operator.}$$

The Spectral Theorem above then gives an orthonormal basis $\{v_n\}_{n\in\mathbb{N}}$ of eigenvectors and eigenvalues $\{\lambda_n\}_{n\in\mathbb{N}}$ for A^{-1}. The third and final observation is that the eigenvector v_n of A^{-1} is also an eigenvector of A, with corresponding eigenvalue[6] λ_n^{-1}. The conclusion the Theorem giving $\lambda_n \to 0$ here yields $\lambda_n^{-1} \to \infty$, truly giving sense to the idea that $A = -\Delta$ is an unbounded operator on $D(A)$.

As a visual summary, consider Figure 1. The dense inclusion $V \hookrightarrow H$ is a continuous, compact embedding of V into H from Theorem 2. Taking duals gives a dense inclusion $H^* \subset V^*$, and identifying H^* with H with Riesz gives the sequence of dense inclusions $V \hookrightarrow H \hookrightarrow V^*$. Lax-Milgram gives the linear isomorphism $A : V \to V^*$, along with its inverse A^{-1}, a compact operator since $V \hookrightarrow H$ is compact.

$$\begin{tikzcd}
V \arrow[r, hook, "\text{compact}"] \arrow[rr, bend left, "A"] & H \arrow[r, hook] & V^* \arrow[ll, bend left, "A^{-1}"]
\end{tikzcd}$$

Figure 1: Sandwich of Hilbert spaces

Finding a basis of eigenvectors allows us to use a *Faedo-Galerkin method*[7]. We will see this more precisely in Theorem 6, but the idea is to project the PDE onto finitely many eigenvectors, turning the problem into systems of ODEs, for which existence and uniqueness is comparatively trivial. As an aside, this method of Galerkin projections is the first step that numerical analysts take to study PDEs. This absolutely crucial step allows a finite machine such as a computer to get a glimpse of what rich behaviour a PDE hides. Of course, to get rigorous results one has to deal with the infinite remainder by pen-and-paper (and brain) analysis [10], but projecting onto finitely many eigenmodes can already capture much of the solution's dynamical properties. A classical example is Lorenz's prototypical chaotic model [5], obtained from projecting an atmospheric convection PDE model onto 3 eigenmodes.

2.5 Weak convergence and compactness

The notion of convergence in Hilbert spaces is trickier than one might initially imagine[8]. The most obvious definition for convergence is to use the given norm, that is, given a sequence $\{u_m\} \subset V$ and some $u \in V$,

[6] One could worry about possible zero eigenvalues, but a zero eigenvalue of A^{-1} would mean non-injectivity, contradicting invertibility.

[7] One can use this method for any separable Hilbert space, but an orthonormal basis of eigenvectors is more tangible than the existence of some orthnormal basis whose existence is given by the ridiculous Axiom of Choice.

[8] It is more involved for general Banach spaces but thankfully we chose $p = 2$ as our integrability exponent, and the Riesz Representation Theorem saves us some trouble.

$u_m \to u$ in V if

$$\|u_m - u\| \to 0, \ n \to \infty.$$

This is known as *strong convergence* and is too strong to expect most times. More specifically, the topology induced by the norm is too restrictive and gives too few compact sets: for example the closed unit ball is not strongly compact. We thereby turn to another notion of convergence using the dual space V^*, known as *weak convergence*, written $u_m \rightharpoonup u$ in V if

$$\langle f, u_m \rangle \to \langle f, u \rangle, \ n \to \infty \ \forall f \in V^*,$$

Theorem 5 (Alaoglu)**.** The closed unit ball $\{u \in V : \|u\| \leq 1\}$ is compact in the weak topology.

With the linear structure of V, Alaoglu's Theorem says any bounded set, contained in some ball of a possibly large radius, is weakly compact.

3 Reaction-diffusion equation with a polynomial nonlinearity

We now have the tools and the language necessary to look at PDEs through a dynamical lens. The setup is as follows: let $\Omega \subset \mathbb{R}^n$ be an open bounded subset with boundary $S = \partial\Omega$ and consider the Cauchy problem for a scalar-valued function $u = u(x,t)$

$$\frac{\partial u}{\partial t} - d\Delta u + g(u) = 0 \qquad \text{for } (x,t) \in \Omega \times \mathbb{R}_+ \tag{2}$$

$$u = 0 \qquad \text{on } S, \tag{3}$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{4}$$

where $\Delta = \Delta_x$ is the Laplacian in the spatial variable x, $d > 0$ is the diffusion coefficient and the reaction term g is an odd degree polynomial with positive leading coefficient:

$$g(s) = \sum_{j=0}^{2p-1} b_j s^j, \ b_{2p-1} > 0.$$

This aims to model chemical reactions, where u is the concentration of some reactant, $-d\Delta u$ represents the diffusion of the reactant throughout the medium and $g(u)$ represents the chemical reaction itself, creating or annihilating the reactant given certain concentrations. Having an odd degree reaction term with a positive leading coefficient means the rate at which the reactant is consumed increases strongly with its concentration.

We now introduce one more bit of notation to study this equation. Since we want to separate the variables into a time variable $t \in \mathbb{R}_+$ and space variables $x \in \Omega$ and apply our dynamical systems lens, we consider (2) as a time-evolution equation for u as a function of x, in the sense that we consider the function

$$t \mapsto u(\cdot, t)$$

as going from some time domain to a space H of functions $u(\cdot,t)$. To this effect, let $L^p(0,T;H)$ denote the space of functions

$$u : t \mapsto u(\cdot,t), \qquad \int_0^T \|u(\cdot,t)\|_H^p \, dt < \infty,$$

where we use the norm in H, or similarly $C(0,T;H)$ denote continuous functions from $[0,T]$ to H, with continuity with respect to the norm in H.

3.1 Existence and Uniqueness

We can now formulate the main result of this section and project, which will take us some steps to prove.

Theorem 6 (Theorem III.1.1 in [9])**.** Let $u_0 \in L^2$, then *there exists a unique weak solution* u of (2)-(4) with

$$u \in L^2(0,T;V) \cap L^{2p}(0,T;L^{2p}), \qquad \forall T > 0, \tag{5}$$

$$u \in C(\mathbb{R}_+, H), \tag{6}$$

where (6) means the mapping $u_0 \mapsto u$ is continuous in H. If we additionally have $u_0 \in H_0^1$, then

$$u \in C([0,T); H_0^1) \cap L^2(0,T;H^2). \tag{7}$$

We break this proof into a few smaller results, the strategy goes as follows: use a Faedo-Galerkin method to get a sequence of approximate solutions, show the sequence of solutions is bounded in appropriate norms, use Alaoglu's Theorem to find a weakly convergent subsequence and finally show the weak limit is a solution to the PDE.

Recalling the discussion in §2.3, consider the bilinear form

$$a(u,v) = d \int_\Omega \nabla u \cdot \nabla v \, dx,$$

and let A be the linear operator associated to a, seen as $A = -d\Delta$. This defines a linear isomorphism $A : V \to V^*$ and can be viewed as a self-adjoint unbounded operator $A : D(A) \to H$ for $D(A) = \{v \in V : Av \in H\}$. The Spectral Theorem gives an orthonormal basis of eigenvectors w_j of A with corresponding eigenvalues. As mentioned in §2.4, we now apply a Faedo-Galerkin method: for every $m \in \mathbb{N}$, let

$$u_m(t) = \sum_{i=1}^{m} f_{mi}(t) w_i \tag{8}$$

be a projection of the solution u onto the first m eigenfunctions, that is

$$\left(\frac{du_m}{dt}, w_j\right) + a(u_m, w_j) + (g(u_m), w_j) = 0, \qquad \text{for } j = 1, \dots, m \tag{9}$$

$$u_m(0) = u_{0m}, \tag{10}$$

where $u_{0m} = \sum_{i=1}^m (u_0, w_i) w_i$ is the projection of the initial condition u_0 onto the subspace spanned by the first m eigenfunctions. The functions f_{mi} are then solutions to an ODE, for which existence and uniqueness is trivial compared to PDEs.

Proposition 2. *For every* $m = 1, 2, \dots,$ *there exists* $T_m > 0$ *and a unique solution* u_m *of the form* (8)*.*

Proof. We can explicitly write out (9) as

$$\sum_{i=1}^{m} f'_{mi}(t)(w_i, w_j) + \sum_{i=1}^{m} f_{mi}(t) a(w_i, w_j) + \left(g\left(\sum_{i=1}^{m} f_{mi}(t) w_i\right), w_j\right) = 0,$$

$$f'_{mj}(t) + \lambda_j f_{mj}(t) + \left(g\left(\sum_{i=1}^{m} f_m(t) w_i\right), w_j\right) = 0$$

to see it is clearly a smooth system of ODEs. The result follows by the standard theorem of existence and uniqueness for ODEs, c.f. [1] for example. □

Before moving onto the energy estimates, we need an analytical lemma dealing with the nonlinearity g. The proof is not particularly enlightening and is left to Appendix C.

Lemma 3. There exist constants $c_2 > 0$ and $c_3 > 0$ such that

$$\frac{1}{2} b_{2p-1} s^{2p} - c_2 \le g(s)s \le \frac{3}{2} b_{2p-1} s^{2p} + c_2 \tag{11}$$

$$\frac{1}{2}(2p-1) b_{2p-1} s^{2p-2} - c_3 \le g'(s) \le \frac{3}{2}(2p-1) b_{2p-1} s^{2p-2} + c_3 \tag{12}$$

for all $s \in \mathbb{R}$.

Proposition 3. *Let $u \in H$ be a solution to the Cauchy problem (2)-(4), we have the bound*

$$|u(t)|^2 \leq |u_0|^2 \exp\left(-\frac{2d}{c_0^2}t\right) + \frac{c_2|\Omega|}{d}\left(1 - \exp\left(-\frac{2d}{c_0^2}t\right)\right), \tag{13}$$

where $c_0 = c_0(\Omega)$ comes from Poincaré's inequality Prop. 1.

Proof. We take $u \in C^\infty(\Omega)$ and extend the result to H by density. We first multiply the PDE (2) by u and integrate over Ω:

$$\begin{aligned}
0 &= \int_\Omega \frac{\partial u}{\partial t} u\, dx - d\int_\Omega u\Delta u\, dx + \int_\Omega g(u)u\, dx \\
&= (\text{using Green's identity and } u = 0 \text{ on } S) \\
&= \frac{1}{2}\frac{d}{dt}|u|^2 + d\int_\Omega \nabla u \cdot \nabla u\, dx + \int_\Omega g(u)u\, dx \\
&\geq (\text{by the lower bound in (11) and the definition of the } V\text{-norm } \|\cdot\|) \\
&\geq \frac{1}{2}\frac{d}{dt}|u|^2 + d\|u\|^2 + \frac{1}{2}b_{2p-1}\int_\Omega u^{2p}\, dx - c_2|\Omega| \\
&\geq (\text{using Poincaré and } b_{2p-1} > 0) \\
&\geq \frac{1}{2}\frac{d}{dt}|u|^2 + \frac{d}{c_0^2}|u|^2 - c_2|\Omega|.
\end{aligned}$$

We obtain the result by applying the classical Grönwall bound in Lemma 1 to

$$\frac{d}{dt}|u(t)|^2 \leq -\frac{2d}{c_0^2}|u(t)|^2 + 2c_2|\Omega|.$$

□

Recall that Prop. 2 only gave us some finite time $T_m > 0$ of existence for all m, but this result improves on that. Indeed, all the approximate solutions u_m obtained are in H, so for each m, applying the above Proposition gives boundedness of $|u_m(t)|$ for all t, meaning that we have global existence and we can take T_m arbitrarily large for all m.

Proposition 4. *Let $u \in V$ be a solution to the Cauchy problem (2)-(4) with $u_0 \in V$, and let $t_0, r > 0$ be arbitrary constants. Then we have the bounds*

$$\begin{aligned}
\|u(t)\|^2 &\leq \|u_0\|^2 \exp(2c_3 t) \quad \text{for } t > 0, \\
\|u(t+r)\|^2 &\leq \frac{\kappa}{r}\exp(2c_3 r) \quad \text{for } t \geq t_0,
\end{aligned}$$

for some $\kappa > 0$.

Proof. Similarly to the proof of Prop. 3, we consider $u \in C_c^\infty$ for computations and argue by density to extend the result to V. We multiply the PDE (2) by $-\Delta u$ and integrate over Ω:

$$\begin{aligned}
0 &= -\int_\Omega \frac{\partial u}{\partial t}\Delta u\, dx + d\int_\Omega |\Delta u|^2\, dx - \int_\Omega g(u)\Delta u\, dx \\
&= (\text{integrating by parts and using the chain rule for } g) \\
&= \sum_{i=1}^n \int_\Omega \frac{\partial^2 u}{\partial x_i \partial t}\frac{\partial u}{\partial x_i}\, dx + d|\Delta u|^2 + \sum_{i=1}^n \int_\Omega g'(u)\left(\frac{\partial u}{\partial x_i}\right)^2\, dx \\
&= \frac{1}{2}\frac{d}{dt}\|u\|^2 + d|\Delta u|^2 + \int_\Omega g'(u)|\nabla u|^2\, dx.
\end{aligned}$$

Now recall from the short discussion following Poincaré's inequality Prop. 1 that $\|u\| \leq c_1|\Delta u|$ for some constant $c_1 = c_1(\Omega) > 0$, and we also use the lower bound in (12) to get

$$\begin{aligned} 0 &\geq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \frac{d}{c_1^2}\|u\|^2 + \frac{1}{2}b_{2p-1}\int_\Omega u^{2p-2}|\nabla u|^2\,dx - c_3\|u\|^2 \\ &\geq \frac{1}{2}\frac{d}{dt}\|u\|^2 - c_3\|u\|^2. \end{aligned}$$

Applying Grönwall's classical bound in Lemma 1 gives the first result. To apply the uniform bound in Lemma 2 and obtain the second result, we take $y(t) = \|u(t)\|^2$, $g(s) = 2c_3$ and $h(s) = 0$. The integrability conditions for g, h are trivial, while for $\|u\|^2$ we need to check

$$\int_t^{t+r} \|u(s)\|^2\,d \leq \kappa$$

for some $\kappa > 0$. For this recall the bound

$$\frac{1}{2}\frac{d}{dt}|u|^2 + d\|u\|^2 + \frac{1}{2}b_{2p-1}\int_\Omega u^{2p}\,dx - c_2|\Omega| \leq 0$$

from the proof of Prop. 3. Discarding the positive $\frac{1}{2}b_{2p-1}\int_\Omega u^{2p}$ term and rearranging gives

$$d\|u\|^2 \leq -\frac{1}{2}\frac{d}{dt}|u|^2 + c_2|\Omega|.$$

Integrating from $s = t$ to $t + r$ gives

$$d\int_t^{t+r} \|u(s)\|^2\,ds \leq \frac{1}{2}(|u(t)|^2 - |u(t+r)|^2) + c_1 r|\Omega| \leq \frac{1}{2}|u(t)|^2 + c_2 r|\Omega|,$$

a bounded quantity since the H-norm is bounded for solutions to (2) by Prop. 3. We can then apply the Uniform Grönwall Lemma and the result follows. □

We can now tackle the proof of the Theorem of Existence and Uniqueness:

Proof of Thm. 6. For each m, let u_m be the unique solution of the form (8), obtained from Galerkin projections. Note the maximal time of existence T_m is $+\infty$ for all m, given our energy estimates Prop. 3 and Prop. 4, so that we can take $T > 0$ arbitrary as the domain of each u_m. More importantly, our estimates give that $\{u_m\}$ is bounded independently of m in all of

$$L^\infty(0, T; H), \quad L^2(0, T; V), \quad L^{2p}(0, T; L^{2p}),$$

so by Alaoglu's Theorem there exists a weakly convergent subsequence which we still denote u_m and some u so that

$$\begin{aligned} &u_m \rightharpoonup u \text{ in } L^2(0, T; V) \text{ and } L^{2p}(0, T; L^{2p}), \\ &u_m \rightharpoonup^* u \text{ in } L^\infty(0, T; H). \end{aligned}$$

Note that the dual space of L^∞ is not an L^p space, hence the need for weak-$*$ convergence, whereas weak convergence for the other spaces is enough by Riesz Representation. We can then take the limit $m \to \infty$ in the finite dimensional ODE (9) to obtain

$$\frac{d}{dt}(u, v) + a(u, v) + (g(u), v) = 0, \quad \forall v \in V \cap L^{2p}.$$

Here we take $v \in L^{2p}$ for the inner product/duality pairing $(g(u), v)$ to make sense: indeed $u(t) \in L^{2p}$ for all t and since $g(s) = O(s^{2p-1})$, we have $g(u(t)) \in L^q$ for $q = 2p/(2p-1)$, which happens to be the conjugate exponent of $2p$. The Riesz Representation Theorem tells us we need to take v in the dual space for the duality pairing to make sense, hence $v \in L^{2p}$. The weak convergence going through $\frac{d}{dt}$ and $a(\cdot, \cdot)$ is trivial with linearity, however dealing with the non-linear term's convergence $g(u_m) \rightharpoonup g(u)$ is more involved, bounding $g(u_m)$ and using weak compactness to get convergence of a subsequence, cf. [4] or [7, Chap. 8]. Then

$$\frac{du}{dt} + Au + g(u) = 0 \text{ in } L^2(0,T;V) \cap L^q(0,T;L^q),$$

where $q = 2p/(2p-1)$ is again the conjugate exponent to $2p$. Uniqueness and continuity on the initial conditions u_0 follow from a modification of the energy estimates. Continuity of $u : [0,T] \to V$ uses Lemma II.3.2 in [9] and the higher regularity results are derived with modifications to our energy estimates, both omitted here. □

We have shown existence of a weak solution to the PDE (2) and swept the details of uniqueness and increasing regularity under the rug. Uniqueness would come from making energy estimates of the difference of 2 functions, obtaining some Grönwall-type bound. The other proofs involve variations on our energy estimates, for example bounding the sequence $\{u_m\}_{m\in\mathbb{N}}$ in higher order Sobolev spaces to obtain higher regularity. Lemma II.3.2 in [9] is a statement about being equal to a continuous function almost everywhere, the proof of which is a couple pages of classic PDEs functional analysis, bounding sequence with certain norms and proving almost-everywhere equalities [8, Chap. III Lemma 1.2]. We need this for our main conclusion of continuity of the semigroup operator $S(t)$, since it allows up to truly take our dynamical approach, but we do not delve into the details.

3.2 Finding an attractor

We claim the PDE (2) has a bounded global attractor, meaning all solutions go inside some bounded absorbing set, but why? The heuristics are rather simple, consider both differential equations

$$\partial_t u + \Delta u = 0, \quad \partial_t u = -g(u).$$

The former is the heat equation, smoothing solutions hence bounding higher order Sobolev norms, while the latter is really just a 1-dimensional ODE, warranting a proper look at the nonlinearity $-g(u)$. We chose the leading coefficient b_{2p-1} of the odd degree polynomial g to be positive, so that $-g(u) \to \mp\infty$ as $u \pm \infty$. Therefore $u \gg 0 \implies \partial_t u \ll 0$ and $u \ll 0 \implies \partial_t u \gg 0$, so that solutions to $\partial_t u = -g(u)$ rapidly decay to some bounded set around the origin, see Figure 2 for a visual version of that explanation. We therefore have a PDE that brings solutions into a bounded set, simultaneously with respect to the usual L^2-norm due to the nonlinearity, as well as higher order Sobolev norms from the heat operator.

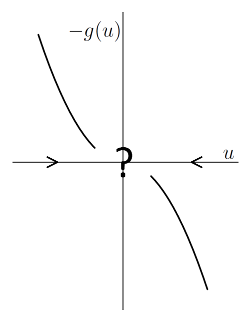

Figure 2: Behaviour of solutions to $\partial_t u = -g(u)$.

Having set our objective, let us see the ingredients necessary to prove existence of an attractor. The first is obtaining an appropriate absorbing set, for which we can look for in either H or V. In fact most of the work was done while obtaining the energy estimates: in H, recall we had

$$\frac{1}{2}\frac{d}{dt}|u|^2 + \frac{d}{c_0^2}|u|^2 - c_2|\Omega| \leq 0,$$

so the classical Grönwall bound is

$$|u(t)|^2 \leq |u_0|^2 \exp\left(-\frac{2d}{c_0^2}t\right) + \frac{c_2 c_0^2|\Omega|}{d}\left(1 - \exp\left(-\frac{2d}{c_0^2}t\right)\right).$$

We then have

$$\limsup_{t\to\infty} |u(t)| \leq \rho_H \text{ for } \rho_H = \frac{c_2 c_0^2|\Omega|}{d}.$$

This doesn't quite define an absorbing set, for this let $\rho'_H > \rho_H$ be arbitrary and take some $R \geq \rho'_H$. We aim to flow solutions from the H-ball $B_R(0)$ for enough time for them to enter the smaller ball $B_{\rho'_H}(0)$: letting $\alpha = -2dt/c_0^2$, we have

$$\begin{aligned} |u(t)|^2 &\leq R^2 e^\alpha + \rho_H^2(1 - e^\alpha) \\ &\leq R^2 e^\alpha + \rho_H^2. \end{aligned}$$

For this to be less than or equal to $(\rho'_H)^2$, we require $\alpha \leq \log\left(\frac{(\rho'_H)^2 - \rho_H^2}{R^2}\right)$, which is if and only if $t \geq t_0$ for

$$t_0 = \frac{2d}{c_0^2} \log\left(\frac{R^2}{(\rho'_H)^2 - \rho_H^2}\right),$$

and the balls $B_{\rho'_H}(0)$ are absorbing in H for all $\rho'_H > \rho_H$. As can be expected the time for entering the ρ'_H-ball increases when R increases, while it decreases when increasing ρ'_H, thus increasing the size of the absorbing set. In fact this dependence on the size of the initial data can be circumvented for this specific form of fast-decaying nonlinearity, cf. Remark 1.3 in [9, §III.1].

We now look for an absorbing ball in V, which will absorb any bounded ball in H. As for the H case, most of the work was already done when deriving the energy estimates. Indeed, for some fixed $r > 0$, Grönwall's uniform lemma gave

$$\|u(t)\|^2 \leq \frac{\kappa}{r} \exp(2c_3 r) \text{ for } t \geq t_0 + r,$$

provided that

$$\int_t^{t+r} \|u(s)\|^2 \, d \leq \kappa,$$

and we obtained the bound

$$d \int_t^{t+r} \|u(s)\|^2 \, ds \leq \frac{1}{2}|u(t)|^2 + c_2 r|\Omega| \text{ for } t \geq t_0.$$

So taking an initial condition $u_0 \in B_R(0)$, the H-ball of radius R at 0, the solution $u(t)$ will be absorbed by the ball of radius ρ'_H, so that

$$\int_t^{t+r} \|u(s)\|^2 \, ds \leq \kappa := \frac{1}{d}\left(\frac{1}{2}(\rho'_H)^2 + c_2 r|\Omega|\right).$$

Taking ρ_V with

$$\rho_V^2 = \frac{\kappa}{r} \exp(2c_3 r)$$

then gives an absorbing ball $B_{\rho_V}(0)$ in V. One thing worth noting, although we leave this to [7, p. 288] for the proof, is that this absorbing ball $B_{\rho_V}(0)$ attracts all bounded sets in V, but also in H. The time evolution of the system thereby not only reduces energy in terms of the H-norm, but also gives bounded V-norm, hence smoothing out the solution in some sense. This should not come as a surprise since the PDE (2) contains a heat operator with a smoothing effect.

The final ingredient required to use Thm. 1 for existence of an attractor is uniform compactness of the semigroup operator $S(t)$ in H, which is now practically trivial. Indeed, any bounded set in H gets absorbed by the ball $B_{\rho_V}(0)$ in V, and by compactness of the embedding $V \hookrightarrow H$ this ball is relatively compact in H.

Theorem 7. The reaction-diffusion PDE (2) has a maximal attractor $\mathcal{A}$, bounded in $V = H_0^1(\Omega)$, compact and connected in $H = L^2(\Omega)$.

3.3 Dynamics on the attractor

All solutions tend toward the attractor $\mathcal{A}$, but precisely how does the attractor influence the long-term dynamics, and how do solutions directly on the attractor behave? We will answer these questions because they are a wonderful finale to end on, but will only do so in expository knowledge, proofs and more in depth discussion can be found in Sections 10–16 of [7].

For large times we know solutions are close to the global attractor $\mathcal{A}$, but this proposition additionally explicitly gives the "shadowing" behaviour that general long-term solutions have in relation to solutions on $\mathcal{A}$.

Proposition 5 (Prop. 10.14 in [7])**.** *Given a trajectory $u(t) = S(t)u_0$, for given $\varepsilon > 0$, $T > 0$, there exists a time $\tau = \tau(\varepsilon, T) > 0$ and a point $v_0 \in \mathcal{A}$ on the attractor such that*

$$|u(\tau + t) - S(t)v_0| \leq \varepsilon \quad \text{for } 0 \leq t \leq T.$$

The trajectory $u(\tau + t)$ is a shifted orbit of $u(t)$: this proposition gives that long-term behaviour of solutions is exactly dictated by solutions on the attractor, and this for any finite time $T > 0$ and allowable error $\varepsilon > 0$. This result should not be surprising, in fact its proof requires no fancy tricks, only continuity on initial conditions and the global attracting property of $\mathcal{A}$.

To help us navigate solutions directly on the attractor, the following theorem gives us a quite remarkable property of the attractor.

Theorem 8 (Thm. 11.11 in [7])**.** The semigroup operators $S(t)$ associated to the PDE (2) satisfy the injectivity property on the attractor $\mathcal{A}$:

$$\text{if there exists } t > 0 \text{ such that } S(t)u_0 = S(t)v_0 \text{ for } u_0, v_0 \in \mathcal{A}, \text{ then } u_0 = v_0.$$

The dissipativity of the PDE gives us existence of solutions for all positive times since they all converge to the attractor, but on the attractor we even have *backwards existence*: invariance of $\mathcal{A}$ under $S(t)$ gives surjectivity of the semigroup on the attractor, and with the result above giving injectivity, we can invert $S(t)$ for all $t \geq 0$, so that we can go in backwards time. In the realm of PDEs, finding existence and uniquness of a solution for all positive times can already be challenging, so having existence and uniqueness that is truly global in time is quite special. Not only do solutions on the attractor exist for all time, but in fact $\mathcal{A}$ is precisely all complete bounded orbits [7, Thm. 10.7], where by complete we mean solutions existing for all positive and negative times.

The attractor enjoys some very interesting structure, which we will analyze using *Lyapunov functionals*, the existence of which give dynamics similar to that of gradient systems where the analysis is reduced to looking at equilibrium points and their stability.

Definition 5. A Lyapunov functional for $S(t)$ on a positively invariant set $X \subset H$, $S(t)X = X$ for all $t \geq 0$, is a continuous functional $\Phi : X \to \mathbb{R}$ such that

1. the function $t \mapsto \Phi(S(t)u_0)$ is non-increasing for all $u_0 \in X$;
2. if $\Phi(S(\tau)u_0) = \Phi(u_0)$ for some $\tau > 0$, then u_0 is a fixed point of $S(t)$, $S(t)u_0 = u_0$ for all $t \geq 0$.

The intuition behind Lyapunov functions is to think of them as energy in some dissipative physical model, say a pendulum with friction: the friction will cause all solutions to die off to some equilibrium point of the system. More generally, a Lyapunov function simplifies the dynamics to behaving like a gradient system, in which case the analysis would be entirely reduced to studying equilibrium points and orbits between them. This tool is used slightly differently in the case here, giving structure of the more complex attractor. With the intuition of energy in mind, we can obtain a Lyapunov function for the reaction-diffusion PDE (2) by taking

$$\Phi(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - \mathcal{G}(u)\,dx = \frac{1}{2}\|u\|^2 - \int_\Omega \mathcal{G}(u)\,dx,$$

where $\mathcal{G}(s) = \int_0^s g(\sigma)\,d\sigma$. It is straightforward to check this defines a Lyapunov functional.

To formulate the next result, we need one more definition: for a fixed point $z \in H$, we call its *unstable set*

$$W^u(z) = \{u_0 \in H : S(t)u_0 \text{ defined for all } t, S(-t)u_0 \to z \text{ as } t \to \infty\},$$

that is all points going to z in backwards time. Let us denote the set of fixed points of $S(t)$ by $\mathcal{E}$.

Theorem 9 (Thm. 10.13 in [7]). The global attractor has structure

$$\mathcal{A} = W^u(\mathcal{E}).$$

Furthermore, since H is connected and if $\mathcal{E}$ is discrete, then

$$\mathcal{A} = W^u(\mathcal{E}) = \bigcup_{z\in\mathcal{E}} W^u(\mathcal{E})$$

and also

$$\mathcal{A} = W^s(\mathcal{E}) = \bigcup_{z\in\mathcal{E}} W^s(z),$$

where $W^s(z) = \{u \in H : S(t)u \to z \text{ as } t \to \infty\}$ is the stable set of the fixed point z.

The idea is that $W^u(\mathcal{E}) \subset \mathcal{A}$ since any point $u \in W^u(\mathcal{E})$ must be a complete bounded orbit since it goes into the attractor in forward time and towards a fixed point in backwards time, hence $u \in \mathcal{A}$; while proving $\mathcal{A} \subset W^u(\mathcal{E})$ uses the non-decreasing property of the Lyapunov function in a slightly more subtle way [7, Prop. 10.12]. The condition that $\mathcal{E}$ is discrete depends on the choice of nonlinearity, and must be done case by case.

3.4 Finite fractal dimension

One cannot talk about dynamical systems without mentioning the fractal structure of the compact attractors. A proper treatment would require too much setup and essentially a whole other 20 pages, so only a brief overview is discussed here. As Robinson puts it: "the noncompactness of the unit ball in an infinite-dimensional space implies that these attractors have no interior". We therefore expect the attractors to be somehow small, but by what measure can we make this rigorous? We claim finite dimensionality of $\mathcal{A}$, but the notion itself of dimension seems flawed here: how should some "messy" set in an infinite-dimensional space H have finite dimension? One must turn to an alternative view on dimension, less instructed by topological intuition, but rather more geometrical.

The 2 notions of dimension relevant to us are (upper) box-counting — or fractal — dimension d_f and the Hausdorff dimension $d_{\mathcal{H}}$ of the attractor $\mathcal{A}$. Roughly speaking, the box-counting dimension d_f is defined by covering $\mathcal{A}$ with balls of decreasing radius $\varepsilon > 0$, and measuring how many more balls one needs to cover $\mathcal{A}$ as $\varepsilon \to 0$. Alternatively the Hausdorff dimension $d_{\mathcal{H}}$ generalizes the idea that d-dimensional volume really is measured by taking lengths to the d-th power, while now allowing for non-integer values of d. The box-counting dimension is conceptually easier to work with, and in fact it bounds the Hausdorff dimension $d_{\mathcal{H}}(X) \leq d_f(X)$ [7, Cor.13.13], although for reasonably computable sets one can show they are equal. With these new ideas of dimension, let us consider an explicit form of reaction nonlinearity

$$g(u) = \lambda(u^3 - u)$$

for some reaction parameter λ, the PDE (2) one obtains is called the Chaffee-Infante equation.

Theorem 10 (Thm. 13.19 in [7]). The dimension of the global attractor $\mathcal{A}$ for

$$\partial_t u - \Delta u = \lambda(u - u^3), \quad u_S = 0, \quad \Omega \subset \mathbb{R}^n$$

is bounded by $d_f(\mathcal{A}) \leq c\lambda^{n/2}$ for some constant $c > 0$.

As a mathematician, the simply stated fact that the global attractor living in an infinite-dimensional space is finite-dimensional in any sense is already remarkable, but as a person living in a physical world, one can wonder about interpreting this result. Temam explains this: "the observed permanent regime depends on a finite number of degrees of freedom". In fact the chapters 14–16 of [7] makes this intuition rigorous by parametrizing the attractor: mapping it into a finite-dimensional space $\mathbb{R}^k$ for $k \geq 2d_f(\mathcal{A}) + 1$ by some injective map L [7, Thm. 16.2]. Furthermore, one can obtain a discrete-time topological conjugacy result for the attractor $\mathcal{A}$ in the Hilbert space H and its image X in the finite-dimensional space $\mathbb{R}^k$, where taking a time step on $\mathcal{A}$ is topologically conjugate to the action of a continuous map f. This is illustrated in Figure 3.

As a final comment that ties back to my research, this finite dimensionality of the long-term dynamics are in part a reason that finite approximations on a computer can still yield relevant insight into the complexity of a system. Indeed if the dynamics were truly infinite-dimensional, finite-dimensional projections would only give an optimistic viewpoint at best, and blurry and erroneous perspective in a worst case scenario.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "S(T)"] \arrow[d, "L"'] & \mathcal{A} \\ X \arrow[r, "f"'] & X \arrow[u, "L^{-1}"'] \end{tikzcd}$$

Figure 3: $S(T)$ takes a time step of T on $\mathcal{A}$ for some $T > 0$, which is conjugate to the action of a continuous function f on X, under the conjugacy L.

A Notation

The following lists the main notation used throughout these notes. It is not exhaustive but collects the symbols and conventions used most frequently and that could be non-standard.

Ω: An open, bounded subset of $\mathbb{R}^n$ with boundary $S = \partial\Omega$, assumed C^1.

$H^k(\Omega)$: Sobolev space of functions with weak derivatives up to order k in $L^2(\Omega)$.

$H_0^1(\Omega)$: Closure of $C_c^\infty(\Omega)$ in $H^1(\Omega)$; functions in H^1 that "vanish" on S.

V: Shorthand for $H_0^1(\Omega)$.

H: Shorthand for $L^2(\Omega)$.

$(\cdot,\cdot)$: The standard inner product on L^2.

$((\cdot,\cdot))$: The inner product on V, $\sum_{i=1}^n (\partial_i u, \partial_i v)$.

$\langle\cdot,\cdot\rangle$: Duality pairing: $\langle f, u\rangle = f(u)$ for u in some Hilbert space and f in its dual.

$|u|$: The L^2 (or H) norm of u, i.e. $|u| = \|u\|_{L^2}$.

$\|u\|$: The H_0^1 (or V) norm only considering the gradient, i.e. $\|u\| = \|\nabla u\|_{L^2}$.

B Proof of the Grönwall lemmas

Proof of Lemma 1: Classical Grönwall. One simply multiplies the bound by an integrating factor $\mu(t_0, t) = \exp\left(-\int_{t_0}^t g(\tau)\, d\tau\right)$, yielding

$$\frac{d}{ds}\left(y(s)\mu(t_0, s)\right) \le h(s)\mu(t_0, s),$$

where s is just a dummy variable replacing t. Integrating with respect to s between t_0 and t,

$$\begin{aligned} y(t)\mu(t_0,t) - y(t_0)\mu(t_0,t_0) &\le \int_{t_0}^t h(s)\mu(t_0,s)\, ds \\ y(t) \le y(t_0)\mu(t,t_0) &+ \mu(t,t_0)\int_{t_0}^t h(s)\mu(t_0,s)\, ds \end{aligned}$$

since the integrating factor has $\mu(t_0,t)^{-1} = \mu(t,t_0)$. The result follows noting that $\mu(t,t_0)\mu(t_0,s) = \mu(t,s) = \mu(s,t)^{-1}$. $\square$

Proof of Lemma 2: Uniform Grönwall. Let s, t, t_1 be such that $t_0 \le t_1 \le t \le s \le t + r$, then we again use the integrating factor $\mu(t,s) = \exp\left(-\int_t^s g(\tau)\, d\tau\right)$ and multiply the bound to again obtain

$$\frac{d}{ds}\left(y(s)\mu(t_0, s)\right) \le h(s)\mu(t_0, s),$$

again with s replacing t. For this proof however, we note $\mu(t_0, s) \le 1$ since g is positive and $t_0 \le s$, then integrate $\frac{d}{ds}(y(s)\mu(t_0, s)) \le h(s)$ with respect to s from t_1 and $t + r$, yielding

$$\begin{aligned} y(t+r)\mu(t,t+r) - y(t_1)\mu(t,t_1) &\le \int_{t_1}^{t+r} h(s)\,ds \\ y(t+r) &\le \mu(t+r,t)\int_{t_1}^{t+r} h(s)\,ds + y(t_1)\mu(t_1,t)\mu(t,t+r) \\ y(t+r) &\le \exp(a_1)a_2 + y(t_1)\exp(a_1) \end{aligned}$$

where we used the bounds for g and h, and again that $\mu(t, t+r)^{-1} = \mu(t+r, t)$ and $\mu(t_1, t)\mu(t, t+r) = \mu(t_1, t+r)$. The result then follows by integrating with respect to t_1 from t to $t + r$. □

C Proof of Lemma 3

Proof. This will repeatedly use Young's inequality: for $x, y > 0$, $\varepsilon > 0$, $\alpha, \beta \in (1, \infty)$ such that $\alpha^{-1} + \beta^{-1} = 1$,

$$xy \le \varepsilon x^\alpha + c(\varepsilon)y^\beta,$$

for $c(\varepsilon) = \frac{(\varepsilon\alpha)^{-\beta/\alpha}}{\beta}$. We have

$$\left|\sum_{j=0}^{2p-2} b_j s^{j+1}\right| \le \sum_{j=0}^{2p-2} |b_j| \cdot |s|^{j+1}$$

by the triangle inequality. We then let $x_j = |s|^{j+1}, y_j = |b_j|$ and take exponent $\alpha_j = \frac{2p}{j+1}$ along with its conjugate exponent $\beta_j = \frac{2p}{2p-j-1}$, with ε_j still to be chosen. Young's inequality yields

$$|s|^{j+1} \cdot |b_j| \le \varepsilon_j |s|^{(j+1)\cdot\frac{2p}{j+1}} + c(\varepsilon_j)|b_j|^{\frac{2p}{2p-j-1}} = \varepsilon_j s^{2p} + c(\varepsilon_j)|b_j|^{\frac{2p}{2p-j-1}},$$

and summing over j gives

$$\sum_{j=0}^{2p-2} |b_j| \cdot |s|^{j+1} \le \left(\sum_{j=0}^{2p-2} \varepsilon_j\right) s^{2p} + \sum_{j=0}^{2p-2} c(\varepsilon_j)|b_j|^{\frac{2p}{2p-j-1}}.$$

We can now choose $\varepsilon_j = \frac{1}{2(2p-1)} b_{2p-1}$, so that the coefficient of s^{2p} is $\frac{1}{2}b_{2p-1}$ and we take c_2 to be the second term above. Altogether, this gives

$$\left|\sum_{j=0}^{2p-2} b_j s^{j+1}\right| \le \frac{1}{2} b_{2p-1} s^{2p} + c_2,$$

which gives the first result (11) after noting $g(s)s - b_{2p-1}s^{2p} = \sum_{j=0}^{2p-2} b_j s^{j+1}$. The second set of bounds (12) comes from a similar repeated use of Young's inequality. □

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